Introduction

The model

Arrow-Debreu equilibrium

Securities markets

Extensions

Leverage effect and real options

Large horizon

Continuous time consumption/production

Conclusion

Belief Dispersion and Decreasing Returns in the Stock Market and in the Real Economy

Elyès Jouini

Séminaire Bachelier, IHP, November 20, 2020

Introduction

- Shiller (1981), stock market volatility is too high relative to the ex post variability of dividends

Introduction

- Shiller (1981), stock market volatility is too high relative to the ex post variability of dividends
- First explanation: financial leverage (Black, 76, Christie, 82), explains only a small part

Introduction

- Shiller (1981), stock market volatility is too high relative to the ex post variability of dividends
- First explanation: financial leverage (Black, 76, Christie, 82), explains only a small part
- Heterogeneous beliefs (Basak 00, 2005, JN, 07, 11, Bhamra-Uppal, 14, Atmaz-Basak, 18): additional source of risk (excessive volatility, risk premium puzzle)

Introduction

- Shiller (1981), stock market volatility is too high relative to the ex post variability of dividends
- First explanation: financial leverage (Black, 76, Christie, 82), explains only a small part
- Heterogeneous beliefs (Basak 00, 2005, JN, 07, 11, Bhamra-Uppal, 14, Atmaz-Basak, 18): additional source of risk (excessive volatility, risk premium puzzle)
- But exogenous dividend/production process (risk/return trade-off)

Introduction

- Shiller (1981), stock market volatility is too high relative to the ex post variability of dividends
- First explanation: financial leverage (Black, 76, Christie, 82), explains only a small part
- Heterogeneous beliefs (Basak 00, 2005, JN, 07, 11, Bhamra-Uppal, 14, Atmaz-Basak, 18): additional source of risk (excessive volatility, risk premium puzzle)
- But exogenous dividend/production process (risk/return trade-off)
- "Although many interesting and useful results emerge from the analysis of exchange of fixed quantities of risk and return, an equally important set of issues arises in connection with the fact that the firm may vary the risk-return combination it offers" Greenberg et al. (78)

In this paper

- Financial markets equilibrium model, finite horizon, continuous time, one consumption good

In this paper

- Financial markets equilibrium model, finite horizon, continuous time, one consumption good
- Large economy : continuum of investors

In this paper

- Financial markets equilibrium model, finite horizon, continuous time, one consumption good
- Large economy : continuum of investors
  - parsimonious description of beliefs dispersion as in AB18

In this paper

- Financial markets equilibrium model, finite horizon, continuous time, one consumption good
- Large economy : continuum of investors
    - parsimonious description of beliefs dispersion as in AB18
    - non-vanishing beliefs dispersion : no investor dominates the economy in relatively extreme states

In this paper

- Financial markets equilibrium model, finite horizon, continuous time, one consumption good
- Large economy : continuum of investors
  - parsimonious description of beliefs dispersion as in AB18
  - non-vanishing beliefs dispersion : no investor dominates the economy in relatively extreme states
- One firm (aggregated representation of the economy)

In this paper

- Financial markets equilibrium model, finite horizon, continuous time, one consumption good
- Large economy : continuum of investors
  - parsimonious description of beliefs dispersion as in AB18
  - non-vanishing beliefs dispersion : no investor dominates the economy in relatively extreme states
- One firm (aggregated representation of the economy)
- Approach à la Greenberg et al. (78) : the firm chooses among a set of risk-return combinations ("technology" for the production of risk and return) *vs* neoclassical approach (capital, labor, investment,...)

In this paper

- Financial markets equilibrium model, finite horizon, continuous time, one consumption good
- Large economy : continuum of investors
  - parsimonious description of beliefs dispersion as in AB18
  - non-vanishing beliefs dispersion : no investor dominates the economy in relatively extreme states
- One firm (aggregated representation of the economy)
- Approach à la Greenberg et al. (78) : the firm chooses among a set of risk-return combinations ("technology" for the production of risk and return) *vs* neoclassical approach (capital, labor, investment,...)
- Capture the possibility for the firm/economy to expand during good times and to contract during bad times.

Our main results

- Existence and full characterization of Arrow-Debreu and Arrow-Radner equilibria

Our main results

- Existence and full characterization of Arrow-Debreu and Arrow-Radner equilibria
  - volatile risk exposure (expands/contracts with good/bad news), quadratic relationship between risk exposure and production level

Our main results

- Existence and full characterization of Arrow-Debreu and Arrow-Radner equilibria
  - volatile risk exposure (expands/contracts with good/bad news), quadratic relationship between risk exposure and production level
  - constant returns to scale : risk adjustments couterbalance beliefs heterogeneity impact

Our main results

- Existence and full characterization of Arrow-Debreu and Arrow-Radner equilibria
  - volatile risk exposure (expands/contracts with good/bad news), quadratic relationship between risk exposure and production level
  - constant returns to scale : risk adjustments couterbalance beliefs heterogeneity impact
  - decreasing returns to scale : risk premium fluctuations, risk premium volatility, and excess volatility

Our main results

- Existence and full characterization of Arrow-Debreu and Arrow-Radner equilibria
  - volatile risk exposure (expands/contracts with good/bad news), quadratic relationship between risk exposure and production level
  - constant returns to scale : risk adjustments couterbalance beliefs heterogeneity impact
  - decreasing returns to scale : risk premium fluctuations, risk premium volatility, and excess volatility
- Due to risk exposure fluctuations

Our main results

- Existence and full characterization of Arrow-Debreu and Arrow-Radner equilibria
  - volatile risk exposure (expands/contracts with good/bad news), quadratic relationship between risk exposure and production level
  - constant returns to scale : risk adjustments couterbalance beliefs heterogeneity impact
  - decreasing returns to scale : risk premium fluctuations, risk premium volatility, and excess volatility
- Due to risk exposure fluctuations
  - skewness, excess kurtosis and momentum in the production process

Our main results

- Existence and full characterization of Arrow-Debreu and Arrow-Radner equilibria
  - volatile risk exposure (expands/contracts with good/bad news), quadratic relationship between risk exposure and production level
  - constant returns to scale : risk adjustments couterbalance beliefs heterogeneity impact
  - decreasing returns to scale : risk premium fluctuations, risk premium volatility, and excess volatility
- Due to risk exposure fluctuations
  - skewness, excess kurtosis and momentum in the production process
  - skewness, excess kurtosis, short-term momentum and long-term reversal in the asset price process

Other features

- Relative disagr. highest during recessions (Bloom, 14)

Other features

- Relative disagr. highest during recessions (Bloom, 14)
- RP $\searrow$ with stock prices (Campbell-Cochrane, 99)

Other features

- Relative disagr. highest during recessions (Bloom, 14)
- RP $\searrow$ with stock prices (Campbell-Cochrane, 99)
- Stock volatility follows a Heston-like behavior

Other features

- Relative disagr. highest during recessions (Bloom, 14)
- RP $\searrow$ with stock prices (Campbell-Cochrane, 99)
- Stock volatility follows a Heston-like behavior
- Volatility risk premium (Carr and Wu, 09), and the RP by unit of risk explodes in bad times (Corradi et al.,13)

Other features

- Relative disagr. highest during recessions (Bloom, 14)
- RP $\searrow$ with stock prices (Campbell-Cochrane, 99)
- Stock volatility follows a Heston-like behavior
- Volatility risk premium (Carr and Wu, 09), and the RP by unit of risk explodes in bad times (Corradi et al.,13)
- Volatility $\searrow$ with returns at the aggregate level (e.g., Schwert-Stambaugh, 87) but $\nearrow$ with returns at the individual firm level (Duffee, 95)

Other features

- Relative disagr. highest during recessions (Bloom, 14)
- RP $\searrow$ with stock prices (Campbell-Cochrane, 99)
- Stock volatility follows a Heston-like behavior
- Volatility risk premium (Carr and Wu, 09), and the RP by unit of risk explodes in bad times (Corradi et al.,13)
- Volatility $\searrow$ with returns at the aggregate level (e.g., Schwert-Stambaugh, 87) but $\nearrow$ with returns at the individual firm level (Duffee, 95)
- Testable conseq. : Sharpe and volatilities ratios bounds and relation between fin. volatility, RP, macro volatility, risk aversion and instantaneous average belief

Other features

- Relative disagr. highest during recessions (Bloom, 14)
- RP $\searrow$ with stock prices (Campbell-Cochrane, 99)
- Stock volatility follows a Heston-like behavior
- Volatility risk premium (Carr and Wu, 09), and the RP by unit of risk explodes in bad times (Corradi et al.,13)
- Volatility $\searrow$ with returns at the aggregate level (e.g., Schwert-Stambaugh, 87) but $\nearrow$ with returns at the individual firm level (Duffee, 95)
- Testable conseq. : Sharpe and volatilities ratios bounds and relation between fin. volatility, RP, macro volatility, risk aversion and instantaneous average belief
- Continuous time and log utility : CIR++ interest rate model

The firm

- $\mathbb{T} = [0, T]$, Brownian motion $(W_t)_{t \in \mathbb{T}}$

The firm

- $\mathbb{T} = [0, T]$, Brownian motion $(W_t)_{t \in \mathbb{T}}$
- One consumption good produced by the firm, delivered at date T, and consumption at T

The firm

- $\mathbb{T} = [0, T]$, Brownian motion $(W_t)_{t \in \mathbb{T}}$
- One consumption good produced by the firm, delivered at date T, and consumption at T
- Controled production process

$$dy_t^\theta = m(\theta_t)\, y_t^\theta dt + \theta_t y_t^\theta dW_t, \;\; y_0^\theta = 1$$

The firm

- $\mathbb{T} = [0, T]$, Brownian motion $(W_t)_{t \in \mathbb{T}}$
- One consumption good produced by the firm, delivered at date T, and consumption at T
- Controled production process

$$dy_t^\theta = m(\theta_t)\, y_t^\theta dt + \theta_t y_t^\theta dW_t, \;\; y_0^\theta = 1$$

- $m(\theta) = \mathfrak{a} + \mathfrak{b}\theta - \mathfrak{c}\theta^2$ models the uncertainty/expected growth rate trade-off or how risk is transformed into return

The firm

- $\mathbb{T} = [0, T]$, Brownian motion $(W_t)_{t \in \mathbb{T}}$
- One consumption good produced by the firm, delivered at date T, and consumption at T
- Controled production process

$$dy_t^\theta = m(\theta_t)\, y_t^\theta dt + \theta_t y_t^\theta dW_t, \;\; y_0^\theta = 1$$

- $m(\theta) = \mathfrak{a} + \mathfrak{b}\theta - \mathfrak{c}\theta^2$ models the uncertainty/expected growth rate trade-off or how risk is transformed into return
- A change in units of measure permits to take $\mathfrak{a} = 0$

The firm

- $\mathbb{T} = [0, T]$, Brownian motion $(W_t)_{t \in \mathbb{T}}$
- One consumption good produced by the firm, delivered at date T, and consumption at T
- Controled production process

$$dy_t^\theta = m(\theta_t)\, y_t^\theta dt + \theta_t y_t^\theta dW_t, \;\; y_0^\theta = 1$$

- $m(\theta) = \mathfrak{a} + \mathfrak{b}\theta - \mathfrak{c}\theta^2$ models the uncertainty/expected growth rate trade-off or how risk is transformed into return
- A change in units of measure permits to take $\mathfrak{a} = 0$
- $Y = \{y : y \leq y_T^\theta \text{ for some } \theta\}$

The firm

Introduction

The model

Arrow-Debreu equilibrium

Securities markets

Extensions

Leverage effect and real options

Large horizon

Continuous time consumption/production

Conclusion

- $\mathbb{T} = [0, T]$, Brownian motion $(W_t)_{t \in \mathbb{T}}$
- One consumption good produced by the firm, delivered at date T, and consumption at T
- Controled production process

$$dy_t^\theta = m(\theta_t)\, y_t^\theta dt + \theta_t y_t^\theta dW_t, \;\; y_0^\theta = 1$$

- $m(\theta) = \mathfrak{a} + \mathfrak{b}\theta - \mathfrak{c}\theta^2$ models the uncertainty/expected growth rate trade-off or how risk is transformed into return
- A change in units of measure permits to take $\mathfrak{a} = 0$
- $Y = \{y : y \leq y_T^\theta \text{ for some } \theta\}$
- AD-Prices (SPD) : r.v. p and value of y given by $E[py]$

Consumers/Shareholder

- There is a continuum of consumers who own the firm

Consumers/Shareholder

- There is a continuum of consumers who own the firm
- Disagree about m : a δ-type shareholder believes that m is given by $m_\delta(\theta) = m(\theta) + \delta\theta$ and

$$dy_t = (m(\theta_t) + \delta\theta_t)\, y_t dt + \theta_t y_t dW_t^\delta$$

Consumers/Shareholder

- There is a continuum of consumers who own the firm
- Disagree about m : a δ-type shareholder believes that m is given by $m_\delta(\theta) = m(\theta) + \delta\theta$ and

$$dy_t = (m(\theta_t) + \delta\theta_t)\, y_t dt + \theta_t y_t dW_t^\delta$$

- A δ-type shareholder has a subjective P^δ with $\frac{dP^\delta}{dP} = M_T^\delta = \exp(-\frac{1}{2}\delta^2 T + \delta W_T)$

Consumers/Shareholder

- There is a continuum of consumers who own the firm
- Disagree about m : a δ-type shareholder believes that m is given by $m_\delta(\theta) = m(\theta) + \delta\theta$ and

$$dy_t = (m(\theta_t) + \delta\theta_t)\, y_t dt + \theta_t y_t dW_t^\delta$$

- A δ-type shareholder has a subjective P^δ with $\frac{dP^\delta}{dP} = M_T^\delta = \exp(-\frac{1}{2}\delta^2 T + \delta W_T)$
- δ takes all possible real values

Consumers/Shareholder

- There is a continuum of consumers who own the firm
- Disagree about m : a δ-type shareholder believes that m is given by $m_\delta(\theta) = m(\theta) + \delta\theta$ and

$$dy_t = (m(\theta_t) + \delta\theta_t)\, y_t dt + \theta_t y_t dW_t^\delta$$

- A δ-type shareholder has a subjective P^δ with $\frac{dP^\delta}{dP} = M_T^\delta = \exp(-\frac{1}{2}\delta^2 T + \delta W_T)$
- δ takes all possible real values
- Consumption at T, $U^\delta(c) = E\left[M_T^\delta u(c)\right]$, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$

Consumers/Shareholder

- There is a continuum of consumers who own the firm
- Disagree about m : a δ-type shareholder believes that m is given by $m_\delta(\theta) = m(\theta) + \delta\theta$ and

$$dy_t = (m(\theta_t) + \delta\theta_t)\, y_t dt + \theta_t y_t dW_t^\delta$$

- A δ-type shareholder has a subjective P^δ with $\frac{dP^\delta}{dP} = M_T^\delta = \exp(-\frac{1}{2}\delta^2 T + \delta W_T)$
- δ takes all possible real values
- Consumption at T, $U^\delta(c) = E\left[M_T^\delta u(c)\right]$, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$
- Initial endowment of agent δ : $\nu_\delta \sim \mathcal{N}\left(\delta_0, \omega^2\right)$

Definition, existence and uniqueness

Definition

$\left(\bar{y}, (\bar{c}_\delta)_{\delta\in\mathbb{R}}, \bar{p}\right)$ is an ADPE if 1. $\bar{y} = \arg\max_Y E[\bar{p}y]$,
2. $\bar{c}_\delta = \text{argmax}\, U^\delta(c),\ E[\bar{p}c] \le \nu_\delta E[\bar{p}\bar{y}]$ for all δ, and
3. $\int \bar{c}_\delta d\delta = \bar{y}$.

Theorem

There exists a unique ADPE given by

$$\bar{p} = \bar{y}^{-\gamma} \exp \frac{(k - W_T)^2}{2\varpi^2},\ \bar{\theta}_{T-\tau} = \frac{W_{T-\tau} - \mathfrak{b}\tau - k + \mathfrak{b}\varpi^2}{(2\mathfrak{c}+\gamma)\varpi^2 - \tau(2\mathfrak{c}+1)},$$

$$\bar{c}_\delta = \frac{\varpi \lambda_\delta^{-\frac{1}{\gamma}} \left(M^\delta\right)^{\frac{1}{\gamma}} \bar{p}^{-\frac{1}{\gamma}}}{\sqrt{2\pi\gamma}},\ \lambda_\delta = \exp\left(\frac{(\varpi^2 - T)\,\delta^2}{2} + k\delta\right)$$

$\varpi^2 \searrow$ *with* ω^2 *from* ∞ *(for* $\omega = 0$*) to* $\frac{2\mathfrak{c}+1}{2\mathfrak{c}+\gamma} T$ *(for* $\omega = \infty$*).*

Sketch of the resolution

- FOC : $M^{\delta}\bar{c}_{\delta}^{-\gamma} = \bar{\lambda}_{\delta}\bar{p} \Longrightarrow \bar{c}_{\delta} = \left(M^{\delta}\right)^{\frac{1}{\gamma}}\left(\bar{\lambda}_{\delta}\bar{p}\right)^{-\frac{1}{\gamma}}$ and $\bar{y} = \bar{p}^{-\frac{1}{\gamma}}\int\left(M^{\delta}\right)^{\frac{1}{\gamma}}\bar{\lambda}_{\delta}^{-\frac{1}{\gamma}}d\delta$

Sketch of the resolution

- FOC : $M^\delta \bar{c}_\delta^{-\gamma} = \bar{\lambda}_\delta \bar{p} \Longrightarrow \bar{c}_\delta = \left(M^\delta\right)^{\frac{1}{\gamma}} \left(\bar{\lambda}_\delta \bar{p}\right)^{-\frac{1}{\gamma}}$ and $\bar{y} = \bar{p}^{-\frac{1}{\gamma}} \int \left(M^\delta\right)^{\frac{1}{\gamma}} \bar{\lambda}_\delta^{-\frac{1}{\gamma}} d\delta$
- $\bar{\theta} = \arg\max_\theta E\left[y_T^\theta \bar{y}^{1-\gamma} \left(\int \left(M^\delta\right)^{\frac{1}{\gamma}} \bar{\lambda}_\delta^{-\frac{1}{\gamma}} d\delta\right)^\gamma\right]$

Sketch of the resolution

- FOC : $M^\delta \bar{c}_\delta^{-\gamma} = \bar{\lambda}_\delta \bar{p} \Longrightarrow \bar{c}_\delta = \left(M^\delta\right)^{\frac{1}{\gamma}} \left(\bar{\lambda}_\delta \bar{p}\right)^{-\frac{1}{\gamma}}$ and $\bar{y} = \bar{p}^{-\frac{1}{\gamma}} \int \left(M^\delta\right)^{\frac{1}{\gamma}} \bar{\lambda}_\delta^{-\frac{1}{\gamma}} d\delta$
- $\bar{\theta} = \arg\max_\theta E\left[y_T^\theta \bar{y}^{1-\gamma}\left(\int \left(M^\delta\right)^{\frac{1}{\gamma}} \bar{\lambda}_\delta^{-\frac{1}{\gamma}} d\delta\right)^\gamma\right]$
- We assume $\ln \bar{\lambda}_\delta$ quadratic in δ, i.e. $\bar{\lambda}_\delta$ proportional to $\lambda_\delta = \exp\left(\frac{\left(\varpi^2 - T\right)\delta^2}{2} + k\delta\right)$

Sketch of the resolution

- FOC : $M^{\delta}\bar{c}_{\delta}^{-\gamma} = \bar{\lambda}_{\delta}\bar{p} \Longrightarrow \bar{c}_{\delta} = \left(M^{\delta}\right)^{\frac{1}{\gamma}}\left(\bar{\lambda}_{\delta}\bar{p}\right)^{-\frac{1}{\gamma}}$ and $\bar{y} = \bar{p}^{-\frac{1}{\gamma}} \int \left(M^{\delta}\right)^{\frac{1}{\gamma}} \bar{\lambda}_{\delta}^{-\frac{1}{\gamma}} d\delta$
- $\bar{\theta} = \arg\max_{\theta} E\left[y_T^{\theta}\bar{y}^{1-\gamma}\left(\int \left(M^{\delta}\right)^{\frac{1}{\gamma}} \bar{\lambda}_{\delta}^{-\frac{1}{\gamma}} d\delta\right)^{\gamma}\right]$
- We assume $\ln \bar{\lambda}_{\delta}$ quadratic in δ, i.e. $\bar{\lambda}_{\delta}$ proportional to $\lambda_{\delta} = \exp\left(\frac{\left(\varpi^2 - T\right)\delta^2}{2} + k\delta\right)$
- $z_t^{\theta} = y_t^{\theta}\bar{y}^{-\gamma}$, $\bar{\theta} = \arg\max_{\theta} E\left[q_T z_T^{\theta}\right]$ where $q_T = \left(\int \left(M^{\delta}\right)^{\frac{1}{\gamma}} \bar{\lambda}_{\delta}^{-\frac{1}{\gamma}} d\delta\right)^{\gamma} = \exp\left(\frac{1}{2}\left(\frac{\left(k - W_T\right)^2}{\varpi^2}\right)\right)$

Sketch of the resolution

- FOC : $M^{\delta}\bar{c}_{\delta}^{-\gamma} = \bar{\lambda}_{\delta}\bar{p} \Longrightarrow \bar{c}_{\delta} = \left(M^{\delta}\right)^{\frac{1}{\gamma}}\left(\bar{\lambda}_{\delta}\bar{p}\right)^{-\frac{1}{\gamma}}$ and $\bar{y} = \bar{p}^{-\frac{1}{\gamma}}\int\left(M^{\delta}\right)^{\frac{1}{\gamma}}\bar{\lambda}_{\delta}^{-\frac{1}{\gamma}}d\delta$
- $\bar{\theta} = \arg\max_{\theta} E\left[y_T^{\theta}\bar{y}^{1-\gamma}\left(\int\left(M^{\delta}\right)^{\frac{1}{\gamma}}\bar{\lambda}_{\delta}^{-\frac{1}{\gamma}}d\delta\right)^{\gamma}\right]$
- We assume $\ln\bar{\lambda}_{\delta}$ quadratic in δ, i.e. $\bar{\lambda}_{\delta}$ proportional to $\lambda_{\delta} = \exp\left(\frac{\left(\varpi^2 - T\right)\delta^2}{2} + k\delta\right)$
- $z_t^{\theta} = y_t^{\theta}\bar{y}^{-\gamma}$, $\bar{\theta} = \arg\max_{\theta} E\left[q_T z_T^{\theta}\right]$ where $q_T = \left(\int\left(M^{\delta}\right)^{\frac{1}{\gamma}}\bar{\lambda}_{\delta}^{-\frac{1}{\gamma}}d\delta\right)^{\gamma} = \exp\left(\frac{1}{2}\left(\frac{\left(k - W_T\right)^2}{\varpi^2}\right)\right)$
- We posit a value function of the form $q_t z_t F(t, w)$, $q_t = E_t\left[q_T\right]$

Sketch of the resolution

- FOC : $M^\delta \bar{c}_\delta^{-\gamma} = \bar{\lambda}_\delta \bar{p} \Longrightarrow \bar{c}_\delta = \left(M^\delta\right)^{\frac{1}{\gamma}} \left(\bar{\lambda}_\delta \bar{p}\right)^{-\frac{1}{\gamma}}$ and $\bar{y} = \bar{p}^{-\frac{1}{\gamma}} \int \left(M^\delta\right)^{\frac{1}{\gamma}} \bar{\lambda}_\delta^{-\frac{1}{\gamma}} d\delta$
- $\bar{\theta} = \arg\max_\theta E\left[y_T^\theta \bar{y}^{1-\gamma} \left(\int \left(M^\delta\right)^{\frac{1}{\gamma}} \bar{\lambda}_\delta^{-\frac{1}{\gamma}} d\delta\right)^\gamma\right]$
- We assume $\ln \bar{\lambda}_\delta$ quadratic in δ, i.e. $\bar{\lambda}_\delta$ proportional to $\lambda_\delta = \exp\left(\frac{(\varpi^2 - T)\delta^2}{2} + k\delta\right)$
- $z_t^\theta = y_t^\theta \bar{y}^{-\gamma}$, $\bar{\theta} = \arg\max_\theta E\left[q_T z_T^\theta\right]$ where $q_T = \left(\int \left(M^\delta\right)^{\frac{1}{\gamma}} \bar{\lambda}_\delta^{-\frac{1}{\gamma}} d\delta\right)^\gamma = \exp\left(\frac{1}{2}\left(\frac{(k - W_T)^2}{\varpi^2}\right)\right)$
- We posit a value function of the form $q_t z_t F(t, w)$, $q_t = E_t\left[q_T\right]$
- $0 = F_t + \mu_t^{\bar{\theta}} F + \frac{1}{2} F_{ww} + \sigma_t^q \sigma_t^{\bar{\theta}} F + \sigma_t^{\bar{\theta}} F_w + \sigma_t^q F_w$

Sketch of the resolution

- FOC : $M^\delta \bar{c}_\delta^{-\gamma} = \bar{\lambda}_\delta \bar{p} \Longrightarrow \bar{c}_\delta = \left(M^\delta\right)^{\frac{1}{\gamma}} \left(\bar{\lambda}_\delta \bar{p}\right)^{-\frac{1}{\gamma}}$ and $\bar{y} = \bar{p}^{-\frac{1}{\gamma}} \int \left(M^\delta\right)^{\frac{1}{\gamma}} \bar{\lambda}_\delta^{-\frac{1}{\gamma}} d\delta$
- $\bar{\theta} = \arg\max_\theta E\left[y_T^\theta \bar{y}^{1-\gamma} \left(\int \left(M^\delta\right)^{\frac{1}{\gamma}} \bar{\lambda}_\delta^{-\frac{1}{\gamma}} d\delta\right)^\gamma\right]$
- We assume $\ln \bar{\lambda}_\delta$ quadratic in δ, i.e. $\bar{\lambda}_\delta$ proportional to $\lambda_\delta = \exp\left(\frac{(\varpi^2 - T)\delta^2}{2} + k\delta\right)$
- $z_t^\theta = y_t^\theta \bar{y}^{-\gamma}$, $\bar{\theta} = \arg\max_\theta E\left[q_T z_T^\theta\right]$ where $q_T = \left(\int \left(M^\delta\right)^{\frac{1}{\gamma}} \bar{\lambda}_\delta^{-\frac{1}{\gamma}} d\delta\right)^\gamma = \exp\left(\frac{1}{2}\left(\frac{(k - W_T)^2}{\varpi^2}\right)\right)$
- We posit a value function of the form $q_t z_t F(t, w)$, $q_t = E_t\left[q_T\right]$
- $0 = F_t + \mu_t^{\bar{\theta}} F + \frac{1}{2} F_{ww} + \sigma_t^q \sigma_t^{\bar{\theta}} F + \sigma_t^{\bar{\theta}} F_w + \sigma_t^q F_w$
- $0 = \frac{d}{d\theta}\left(\mu_t^\theta F + \sigma_t^q \sigma_t^\theta F + \sigma_t^\theta F_w\right)\Big|_{\bar{\theta}} = 0$, $1 = F(T, w)$

Sketch of the resolution (continued)

Belief Dispersion and Decreasing Returns

Elyès Jouini

Introduction

The model

Arrow-Debreu equilibrium

Securities markets

Extensions

Leverage effect and real options

Large horizon

Continuous time consumption/production

Conclusion

- $F(t,w)$ of the form $\exp(A(t)w^2 + B(t)w + C(t))$ and $\bar{\theta}(t,w) = D(t)w + E(t)$

Sketch of the resolution (continued)

- $F(t,w)$ of the form $\exp(A(t)w^2 + B(t)w + C(t))$ and $\bar{\theta}(t,w) = D(t)w + E(t)$
- (A, B, C) satisfy a step system of ODEs, Ricatti in A then linear in B then $C \rightarrow \bar{\theta}$

Sketch of the resolution (continued)

- $F(t,w)$ of the form $\exp(A(t)w^2 + B(t)w + C(t))$ and $\bar{\theta}(t,w) = D(t)w + E(t)$
- (A, B, C) satisfy a step system of ODEs, Ricatti in A then linear in B then $C \longrightarrow \bar{\theta}$
- $E\left[\bar{p}\bar{c}_\delta\right] = \vartheta \exp\left(-\frac{1}{2}\frac{\delta^2\varpi^2}{\gamma-1}\right) E\left[\exp\left(\frac{1}{2}\frac{\left(W_T-k-\frac{\varpi^2}{1-\gamma}\delta\right)^2}{\varpi^2\left(1-\frac{1}{\gamma}\right)^{-1}}\right) z_T\right]$ should be proportional to ν_δ

Sketch of the resolution (continued)

- $F(t,w)$ of the form $\exp(A(t)w^2 + B(t)w + C(t))$ and $\bar{\theta}(t,w) = D(t)w + E(t)$
- (A,B,C) satisfy a step system of ODEs, Ricatti in A then linear in B then $C \longrightarrow \bar{\theta}$
- $E\left[\bar{p}\bar{c}_\delta\right] = \vartheta \exp\left(-\frac{1}{2}\frac{\delta^2\varpi^2}{\gamma-1}\right) E\left[\exp\left(\frac{1}{2}\frac{\left(W_T - k - \frac{\varpi^2}{1-\gamma}\delta\right)^2}{\varpi^2\left(1-\frac{1}{\gamma}\right)^{-1}}\right) z_T\right]$ should be proportional to ν_δ
- Gives $k = \frac{\mathfrak{b}(1-\gamma)+\delta_0(2\mathfrak{c}+1)}{2\mathfrak{c}+\gamma} T - \varpi^2\delta_0$ and $\frac{1}{\omega^2} = \frac{\left((2c+\gamma)\varpi^2 + (2c+1)T\right)\varpi^2}{2T\mathfrak{c}(1-\gamma)+\gamma(2\mathfrak{c}+\gamma)\varpi^2}$

Sketch of the resolution (continued)

- $F(t,w)$ of the form $\exp(A(t)w^2 + B(t)w + C(t))$ and $\bar{\theta}(t,w) = D(t)w + E(t)$
- (A, B, C) satisfy a step system of ODEs, Ricatti in A then linear in B then $C \to \bar{\theta}$
- $E\left[\bar{p}\bar{c}_\delta\right] = \vartheta \exp\left(-\frac{1}{2}\frac{\delta^2\varpi^2}{\gamma-1}\right) E\left[\exp\left(\frac{1}{2}\frac{\left(W_T-k-\frac{\varpi^2}{1-\gamma}\delta\right)^2}{\varpi^2\left(1-\frac{1}{\gamma}\right)^{-1}}\right) z_T\right]$ should be proportional to ν_δ
- Gives $k = \frac{\mathfrak{b}(1-\gamma)+\delta_0(2\mathfrak{c}+1)}{2\mathfrak{c}+\gamma} T - \varpi^2\delta_0$ and $\frac{1}{\omega^2} = \frac{\left((2c+\gamma)\varpi^2+(2c+1)T\right)\varpi^2}{2T\mathfrak{c}(1-\gamma)+\gamma(2\mathfrak{c}+\gamma)\varpi^2}$
- Uniqueness : adapt Dana (1995), BDJ (2020)

Homogeneous/Exogenous benchmarks

- Without divergence of opinion ($\omega = 0$, $\varpi = \infty$), $\theta_h \equiv \frac{\mathfrak{b}+\delta_0}{2\mathfrak{c}+\gamma}$

Homogeneous/Exogenous benchmarks

- Without divergence of opinion ($\omega = 0$, $\varpi = \infty$), $\theta_{\mathsf{h}} \equiv \frac{\mathfrak{b}+\delta_0}{2\mathfrak{c}+\gamma}$
- In general, $\bar{\theta}_0 = \theta_{\mathsf{h}}$ but $\bar{\theta}_t$ is stochastic: $\bar{\theta}_t \nearrow$ with W_t, and its sensitivity $\nearrow$ with ω^2; $\bar{\theta}_t \nearrow$ with δ_0

Homogeneous/Exogenous benchmarks

- Without divergence of opinion ($\omega = 0$, $\varpi = \infty$), $\theta_{\mathsf{h}} \equiv \frac{\mathfrak{b}+\delta_0}{2\mathfrak{c}+\gamma}$
- In general, $\bar{\theta}_0 = \theta_{\mathsf{h}}$ but $\bar{\theta}_t$ is stochastic: $\bar{\theta}_t \nearrow$ with W_t, and its sensitivity $\nearrow$ with ω^2; $\bar{\theta}_t \nearrow$ with δ_0
- Exogenous benchmark : let μ and σ be given and let us take $\mathfrak{b} = 2\sigma\mathfrak{c}$, $\mathfrak{a} = \mu - \mathfrak{c}\sigma^2$ and $\mathfrak{c} \to \infty$,

Homogeneous/Exogenous benchmarks

- Without divergence of opinion ($\omega = 0$, $\varpi = \infty$), $\theta_{\mathrm{h}} \equiv \frac{\mathfrak{b}+\delta_0}{2\mathfrak{c}+\gamma}$
- In general, $\bar{\theta}_0 = \theta_{\mathrm{h}}$ but $\bar{\theta}_t$ is stochastic: $\bar{\theta}_t \nearrow$ with W_t, and its sensitivity $\nearrow$ with ω^2; $\bar{\theta}_t \nearrow$ with δ_0
- Exogenous benchmark : let μ and σ be given and let us take $\mathfrak{b} = 2\sigma\mathfrak{c}$, $\mathfrak{a} = \mu - \mathfrak{c}\sigma^2$ and $\mathfrak{c} \to \infty$,
  - we have $\bar{\theta}\,(t, W_t) = \sigma$ and $dy_t = \mu y_t dt + \sigma y_t dW_t$ (Atmaz-Basak, 2018)

Homogeneous/Exogenous benchmarks

- Without divergence of opinion ($\omega = 0$, $\varpi = \infty$), $\theta_{\mathrm{h}} \equiv \frac{\mathfrak{b}+\delta_0}{2\mathfrak{c}+\gamma}$
- In general, $\bar{\theta}_0 = \theta_{\mathrm{h}}$ but $\bar{\theta}_t$ is stochastic: $\bar{\theta}_t \nearrow$ with W_t, and its sensitivity $\nearrow$ with ω^2; $\bar{\theta}_t \nearrow$ with δ_0
- Exogenous benchmark : let μ and σ be given and let us take $\mathfrak{b} = 2\sigma\mathfrak{c}$, $\mathfrak{a} = \mu - \mathfrak{c}\sigma^2$ and $\mathfrak{c} \to \infty$,
  - we have $\bar{\theta}(t, W_t) = \sigma$ and $dy_t = \mu y_t dt + \sigma y_t dW_t$ (Atmaz-Basak, 2018)
  - natural, since $m(\sigma) = \mu$ for all $\mathfrak{c}$ and $\lim_{\mathfrak{c}\to\infty} m(\theta) = -\infty$ for $\theta \neq \sigma$

The production process

- At equilibrium, $\bar{y}_t = Y(t, W_t)$ where

$$\begin{aligned} Y(t,w) &= K_t \exp \frac{1}{2}\left(\varphi(t)\bar{\theta}^2(t,w) - \frac{\left(k - \mathfrak{b}\varpi^2 + T\mathfrak{b}\right)^2}{\varphi(0)}\right) \\ \varphi(t) &= (2\mathfrak{c}+\gamma)\varpi^2 - (2\mathfrak{c}+1)(T-t) \\ \text{and } K_t &= \left(\frac{\varphi(t)}{\varphi(0)}\right)^{-\frac{1}{2(2\mathfrak{c}+1)}} \end{aligned}$$

The production process

- At equilibrium, $\bar{y}_t = Y(t, W_t)$ where

$$
\begin{aligned}
Y(t,w) &= K_t \exp \frac{1}{2}\left(\varphi(t)\bar{\theta}^2(t,w) - \frac{\left(k - \mathfrak{b}\varpi^2 + T\mathfrak{b}\right)^2}{\varphi(0)}\right) \\
\varphi(t) &= (2\mathfrak{c}+\gamma)\varpi^2 - (2\mathfrak{c}+1)(T-t) \\
\text{and } K_t &= \left(\frac{\varphi(t)}{\varphi(0)}\right)^{-\frac{1}{2(2\mathfrak{c}+1)}}
\end{aligned}
$$

- Non Markov

The production process

- At equilibrium, $\bar{y}_t = Y(t, W_t)$ where

$$Y(t,w) = K_t \exp \frac{1}{2}\left(\varphi(t)\bar{\theta}^2(t,w) - \frac{\left(k - \mathfrak{b}\varpi^2 + T\mathfrak{b}\right)^2}{\varphi(0)}\right)$$

$$\varphi(t) = (2\mathfrak{c} + \gamma)\varpi^2 - (2\mathfrak{c}+1)(T-t)$$

$$\text{and } K_t = \left(\frac{\varphi(t)}{\varphi(0)}\right)^{-\frac{1}{2(2\mathfrak{c}+1)}}$$

- Non Markov
- $\ln \bar{y}_{s+t}/\bar{y}_s$: skewness > 0 and excess kurtosis for small t

The production process

- At equilibrium, $\bar{y}_t = Y(t, W_t)$ where

$$
\begin{aligned}
Y(t,w) &= K_t \exp \frac{1}{2}\left(\varphi(t)\bar{\theta}^2(t,w) - \frac{\left(k - \mathfrak{b}\varpi^2 + T\mathfrak{b}\right)^2}{\varphi(0)}\right) \\
\varphi(t) &= (2\mathfrak{c}+\gamma)\varpi^2 - (2\mathfrak{c}+1)(T-t) \\
\text{and } K_t &= \left(\frac{\varphi(t)}{\varphi(0)}\right)^{-\frac{1}{2(2\mathfrak{c}+1)}}
\end{aligned}
$$

- Non Markov
- $\ln \bar{y}_{s+t}/\bar{y}_s$: skewness > 0 and excess kurtosis for small t
- Momentum when $\delta_0 = 0$ (no bias) and $\gamma > 1$ and no reversal (Mao and Wei, 2014)

The production process

- At equilibrium, $\bar{y}_t = Y(t, W_t)$ where

$$Y(t,w) = K_t \exp \frac{1}{2}\left(\varphi(t)\bar{\theta}^2(t,w) - \frac{\left(k - \mathfrak{b}\varpi^2 + T\mathfrak{b}\right)^2}{\varphi(0)}\right)$$

$$\varphi(t) = (2\mathfrak{c}+\gamma)\varpi^2 - (2\mathfrak{c}+1)(T-t)$$

$$\text{and } K_t = \left(\frac{\varphi(t)}{\varphi(0)}\right)^{-\frac{1}{2(2\mathfrak{c}+1)}}$$

- Non Markov
- $\ln \bar{y}_{s+t}/\bar{y}_s$: skewness > 0 and excess kurtosis for small t
- Momentum when $\delta_0 = 0$ (no bias) and $\gamma > 1$ and no reversal (Mao and Wei, 2014)
- No such effects in the homogeneous/exogenous settings

Long-lived securities

- Two long-lived securities, risky stock and riskless bond, $dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$ and $r = 0$
- Self-financing strategy α, $dV_t^{\alpha} = \alpha_t \left(\mu_t S_t dt + \sigma_t S_t dW_t \right)$

Theorem

At the ADPE

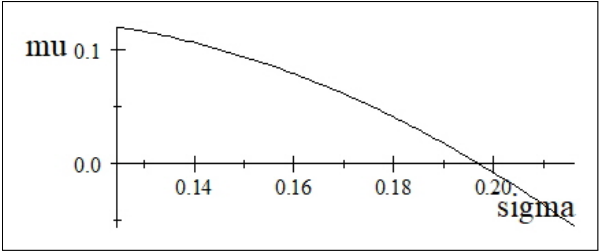
$$\mu_t = \varpi^2 \left(2\mathfrak{c} + \gamma\right) \frac{\mathfrak{b}\varpi^2\gamma + 2\mathfrak{c}k - 2\mathfrak{c}W_t}{\left(\varphi(t) + T - t\right)^2} \bar{\theta}_t,$$

$$\sigma_t = \frac{\left(2\mathfrak{c} + \gamma\right)\varpi^2}{\left(\varphi(t) + T - t\right)} \bar{\theta}_t.$$

For each δ, $\exists \bar{\alpha}^{\delta}$ *s.t.* $V^{\bar{\alpha}^{\delta}} = \bar{c}_{\delta}$, *and* $\int \bar{\alpha}^{\delta} d\delta = 1$.

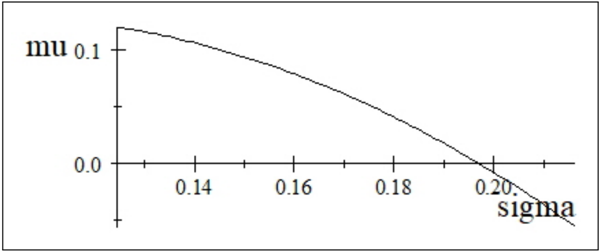
As in Duffie and Huang (1985), the ADPE might be implemented in a Radner equilibrium with two securities.

Drift/RP as a function of volatility (Schwert-Stambaugh 87, Campbell-Hentschel 92)



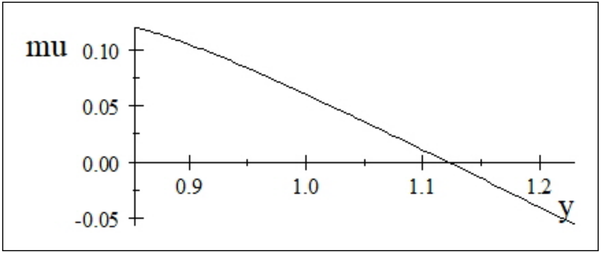
- $\mu \searrow \sigma$

Drift/RP as a function of volatility (Schwert-Stambaugh 87, Campbell-Hentschel 92)

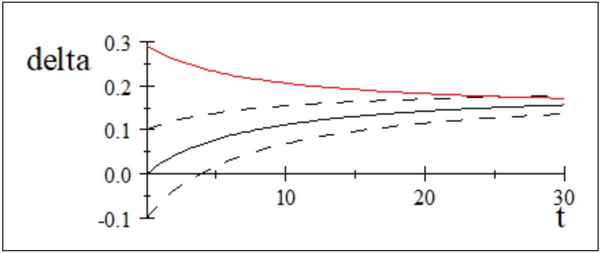


- $\mu \searrow \sigma$
- $\sigma \nearrow$ with θ and with ω

Drift/RP as a function of production level (countercyclical, Campbell-Cochrane, 99)



Evolution of average belief/dispersion



Average belief as a function of production level (procyclical)

Introduction

The model

Arrow-Debreu equilibrium

Securities markets

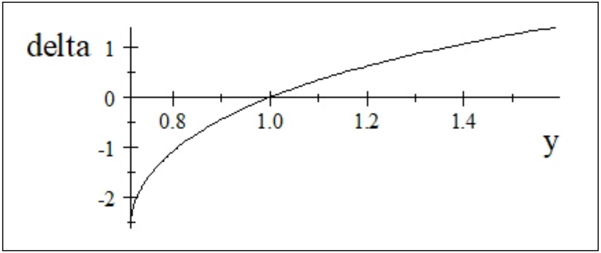
Extensions

Leverage effect and real options

Large horizon

Continuous time consumption/production

Conclusion



Drift, volatilities and disagreement

- $\sigma_t / \bar{\theta}_t \nearrow$ with ω from 1 $(\omega = 0)$ to $\frac{\max(1,\gamma)+2\mathfrak{c}}{\max(1,\gamma)+2\mathfrak{c}t/T}$ $(\omega = \infty)$

Drift, volatilities and disagreement

Belief Dispersion and Decreasing Returns

Elyès Jouini

Introduction

The model

Arrow-Debreu equilibrium

Securities markets

Extensions

Leverage effect and real options

Large horizon

Continuous time consumption/production

Conclusion

- $\sigma_t/\bar{\theta}_t \nearrow$ with ω from 1 ($\omega = 0$) to $\frac{\max(1,\gamma)+2c}{\max(1,\gamma)+2ct/T}$ ($\omega = \infty$)
- $\mu_t = (\gamma - 1)\sigma_t\bar{\theta}_t + \sigma_t^2 - \sigma_t\delta_t$, positively related to σ_t when controling for $\bar{\theta}_t$ (for $\gamma > 1$ and $\delta_t = 0$) Duffee, 1995, (contrast with no control)

Drift, volatilities and disagreement

- $\sigma_t/\bar{\theta}_t \nearrow$ with ω from 1 ($\omega = 0$) to $\frac{\max(1,\gamma)+2\mathfrak{c}}{\max(1,\gamma)+2\mathfrak{c}t/T}$ ($\omega = \infty$)
- $\mu_t = (\gamma-1)\sigma_t\bar{\theta}_t + \sigma_t^2 - \sigma_t\delta_t$, positively related to σ_t when controling for $\bar{\theta}_t$ (for $\gamma > 1$ and $\delta_t = 0$) Duffee, 1995, (contrast with no control)
- $\frac{\sigma_{T-\tau}}{\bar{\theta}_{T-\tau}}$ takes all possible values in

$$\left[1, \frac{2+\sqrt{\left(\tau\omega_{T-\tau}^2-\gamma\right)^2+4\tau\omega_{T-\tau}^2}+\tau\omega_{T-\tau}^2-\gamma}{2}\right]$$

when $\mathfrak{b}$ and $\mathfrak{c}$ take all admissible values.

Volatilities ratio bounds as a function of risk aversion

Introduction

The model

Arrow-Debreu equilibrium

Securities markets

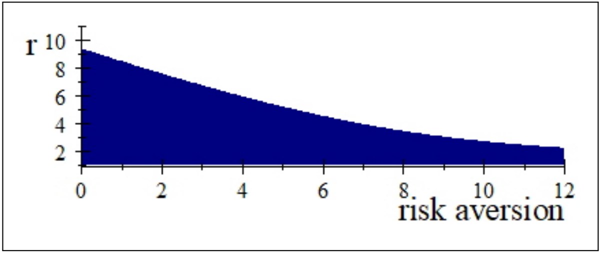
Extensions

Leverage effect and real options

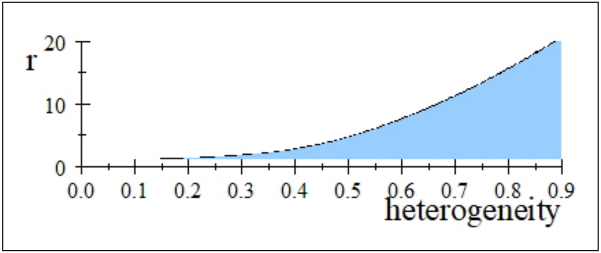
Large horizon

Continuous time consumption/production

Conclusion



Volatilities ratio bounds as a function of disagreement



Volatility risk premium

- $V_t = \sigma_t^2; \; dV_t = \left(D_t^0 + \mathfrak{b} D_t^1 \sqrt{V_t} + D_t^2 V_t\right) dt + D_t \sqrt{V_t} dW$

Volatility risk premium

- $V_t = \sigma_t^2$; $dV_t = \left(D_t^0 + \mathfrak{b} D_t^1 \sqrt{V_t} + D_t^2 V_t\right) dt + D_t \sqrt{V_t} dW$
- Stochastic volatility, $\mathrm{vol} - \mathrm{vol} = D_t$ (in contrast with the exogenous setting)

Volatility risk premium

- $V_t = \sigma_t^2$; $dV_t = \left(D_t^0 + \mathfrak{b} D_t^1 \sqrt{V_t} + D_t^2 V_t\right) dt + D_t \sqrt{V_t} dW$
- Stochastic volatility, $\mathrm{vol} - \mathrm{vol} = D_t$ (in contrast with the exogenous setting)
- For $\mathfrak{b} = 0$, Heston model with perfect correlation between the 2 sources of risk

Volatility risk premium

- $V_t = \sigma_t^2$; $dV_t = \left(D_t^0 + \mathfrak{b} D_t^1 \sqrt{V_t} + D_t^2 V_t\right) dt + D_t \sqrt{V_t} dW$
- Stochastic volatility, $\mathrm{vol} - \mathrm{vol} = D_t$ (in contrast with the exogenous setting)
- For $\mathfrak{b} = 0$, Heston model with perfect correlation between the 2 sources of risk
- The volatility risk premium by unit of risk $\Lambda_t = \frac{\mu_t}{\sigma_t} \frac{D_t}{\sqrt{V_t}} \longrightarrow \infty$ during recessions $(W_t^* = \mathfrak{b}\,(T - t) + k - \mathfrak{b}\varpi^2,\ \bar{\theta}_t = 0,\ S_t$ minimal)

Equilibrium stock price

- $S_t = G(t, \bar{y}_t)$ where $G(t, y) = H(t) y^{K(t)}$ with $K(t) = \frac{\sigma_t}{\bar{\theta}_t}$ and H satisfies a first-order ODE

Equilibrium stock price

- $S_t = G(t, \bar{y}_t)$ where $G(t,y) = H(t)y^{K(t)}$ with $K(t) = \frac{\sigma_t}{\bar{\theta}_t}$ and H satisfies a first-order ODE
- Log-returns exhibit positive skewness and excess kurtosis

Equilibrium stock price

Belief Dispersion and Decreasing Returns

Elyès Jouini

Introduction

The model

Arrow-Debreu equilibrium

Securities markets

Extensions

Leverage effect and real options

Large horizon

Continuous time consumption/production

Conclusion

- $S_t = G(t, \bar{y}_t)$ where $G(t, y) = H(t) y^{K(t)}$ with $K(t) = \frac{\sigma_t}{\bar{\theta}_t}$ and H satisfies a first-order ODE
- Log-returns exhibit positive skewness and excess kurtosis
- $C(S, s, t) = \lim_{h \to 0} \frac{1}{h^2} \mathrm{cov}(\ln \frac{S_{s+h}}{S_s}, \ln \frac{S_{t+h}}{S_t})$

Equilibrium stock price

- $S_t = G(t, \bar{y}_t)$ where $G(t, y) = H(t) y^{K(t)}$ with $K(t) = \frac{\sigma_t}{\bar{\theta}_t}$ and H satisfies a first-order ODE
- Log-returns exhibit positive skewness and excess kurtosis
- $C(S, s, t) = \lim_{h \to 0} \frac{1}{h^2} \mathrm{cov}(\ln \frac{S_{s+h}}{S_s}, \ln \frac{S_{t+h}}{S_t})$
- If $\delta_0 = 0, \gamma > 1$ and ω or T large enough, $C(S, 0, t) > 0$ for t small : short-term momentum

Equilibrium stock price

- $S_t = G(t, \bar{y}_t)$ where $G(t, y) = H(t) y^{K(t)}$ with $K(t) = \frac{\sigma_t}{\bar{\theta}_t}$ and H satisfies a first-order ODE
- Log-returns exhibit positive skewness and excess kurtosis
- $C(S, s, t) = \lim_{h \to 0} \frac{1}{h^2} \text{cov}(\ln \frac{S_{s+h}}{S_s}, \ln \frac{S_{t+h}}{S_t})$
- If $\delta_0 = 0, \gamma > 1$ and ω or T large enough, $C(S, 0, t) > 0$ for t small : short-term momentum
- For T or ω large enough and $\xi \in (0, 1)$, $C(S, 0, \xi T) < 0$: long-run reversal

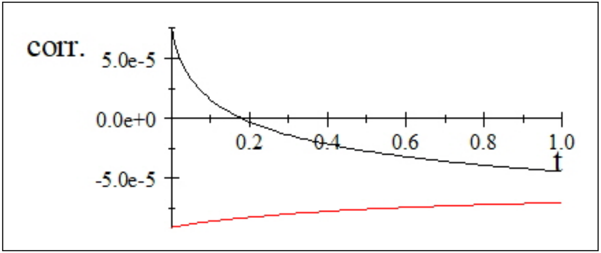
Equilibrium stock price

- $S_t = G(t, \bar{y}_t)$ where $G(t, y) = H(t) y^{K(t)}$ with $K(t) = \frac{\sigma_t}{\bar{\theta}_t}$ and H satisfies a first-order ODE
- Log-returns exhibit positive skewness and excess kurtosis
- $C(S, s, t) = \lim_{h \to 0} \frac{1}{h^2} \mathrm{cov}(\ln \frac{S_{s+h}}{S_s}, \ln \frac{S_{t+h}}{S_t})$
- If $\delta_0 = 0, \gamma > 1$ and ω or T large enough, $C(S, 0, t) > 0$ for t small : short-term momentum
- For T or ω large enough and $\xi \in (0, 1)$, $C(S, 0, \xi T) < 0$: long-run reversal
- In the exogenous setting, short-term and long-run negative autocorrelations of returns

Equilibrium stock price

- $S_t = G(t, \bar{y}_t)$ where $G(t, y) = H(t)y^{K(t)}$ with $K(t) = \frac{\sigma_t}{\bar{\theta}_t}$ and H satisfies a first-order ODE
- Log-returns exhibit positive skewness and excess kurtosis
- $C(S, s, t) = \lim_{h \to 0} \frac{1}{h^2} \mathrm{cov}(\ln \frac{S_{s+h}}{S_s}, \ln \frac{S_{t+h}}{S_t})$
- If $\delta_0 = 0, \gamma > 1$ and ω or T large enough, $C(S, 0, t) > 0$ for t small : short-term momentum
- For T or ω large enough and $\xi \in (0, 1)$, $C(S, 0, \xi T) < 0$: long-run reversal
- In the exogenous setting, short-term and long-run negative autocorrelations of returns
- In the homogeneous framework, no momentum nor long-run reversal

Autocorrelation between date 0 and date t returns



- Debt : the value of the firm at T^* is given by $[S_{T^*} - \Delta]^+$

Belief Dispersion and Decreasing Returns
Elyès Jouini
Introduction
The model
Arrow-Debreu equilibrium
Securities markets
Extensions
Leverage effect and real options
Large horizon
Continuous time consumption/production
Conclusion

- Debt : the value of the firm at T^* is given by $[S_{T^*} - \Delta]^+$
- Call option price with maturity T^* and strike κ

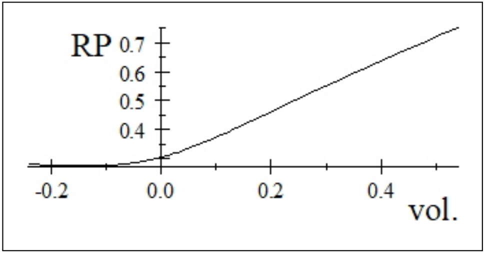
$$C_t^{\kappa, T^*} = S_t \left(\mathcal{N}\left(d_1\right) + \mathcal{N}\left(d_1'\right)\right) - \kappa \left(\mathcal{N}\left(d_2\right) + \mathcal{N}\left(d_2'\right)\right)$$

$$d_1 = \frac{U - P}{\sqrt{Q}}, \; d_1' = \frac{P - V}{\sqrt{Q}}, \; d_2 = \frac{U - p}{\sqrt{q}}, \; d_2' = \frac{p - V}{\sqrt{q}},$$

where $U < V$ are the solutions of $L_2 x^2 + L_1 x = \ln \kappa$.

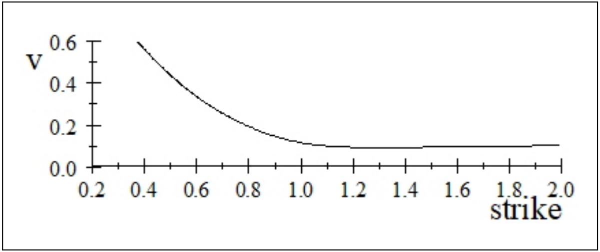
Leverage effect and real options

The risk premium increases with volatility at the idividual level (Duffee, 95)



Leverage effect and real options

Volatility smirk (Rubinstein 94, AïtSahalia-Lo 98, Foresi-Wu 05)



Large horizon

Introduction
The model
Arrow-Debreu equilibrium
Securities markets
Extensions
Leverage effect and real options
Large horizon
Continuous time consumption/production
Conclusion

- Taking the limit when $T \to \infty$, we have

$$\bar{\theta}_t = \frac{\mathfrak{b} + \delta_t}{2\mathfrak{c} + \gamma}, \ \frac{\mu_t}{\sigma_t} = \mathfrak{b}, \ \frac{\sigma_t}{\bar{\theta}_t} = 2\mathfrak{c} + 1, \ \delta_\infty = \mathfrak{b}\frac{\gamma - 1}{2\mathfrak{c} + 1}$$

Large horizon

- Taking the limit when $T \longrightarrow \infty$, we have

$$\bar{\theta}_t = \frac{\mathfrak{b} + \delta_t}{2\mathfrak{c} + \gamma},\ \frac{\mu_t}{\sigma_t} = \mathfrak{b},\ \frac{\sigma_t}{\bar{\theta}_t} = 2\mathfrak{c} + 1,\ \delta_\infty = \mathfrak{b}\frac{\gamma - 1}{2\mathfrak{c} + 1}$$

- $\sigma_{[0,t]} = \sqrt{\frac{1}{t}\mathrm{V}\textsc{ar}\left[\ln S_t\right]}$, $\mu_{[0,t]} = \frac{1}{t}\mathrm{E}\left[\ln S_t\right] + \frac{1}{2}\sigma^2_{[0,t]}$ and $\mathrm{S}\textsc{harpe}_{[0,t]} = \frac{\mu_{[0,t]}}{\sigma_{[0,t]}}$

Large horizon

- Taking the limit when $T \to \infty$, we have

$$\bar{\theta}_t = \frac{\mathfrak{b} + \delta_t}{2\mathfrak{c} + \gamma}, \ \frac{\mu_t}{\sigma_t} = \mathfrak{b}, \ \frac{\sigma_t}{\bar{\theta}_t} = 2\mathfrak{c} + 1, \ \delta_\infty = \mathfrak{b}\frac{\gamma - 1}{2\mathfrak{c} + 1}$$

- $\sigma_{[0,t]} = \sqrt{\frac{1}{t}\mathrm{V{\scriptsize AR}}\left[\ln S_t\right]}$, $\mu_{[0,t]} = \frac{1}{t}\mathrm{E}\left[\ln S_t\right] + \frac{1}{2}\sigma^2_{[0,t]}$ and $\mathrm{S{\scriptsize HARPE}}_{[0,t]} = \frac{\mu_{[0,t]}}{\sigma_{[0,t]}}$
- $\bar{\sigma}_t = \mathrm{E}\left[\sigma_t\right]$, $\bar{\mu}_t = \mathrm{E}\left[\mu_t\right]$ and $\mathrm{S{\scriptsize HARPE}}_t = \frac{\mu_t}{\sigma_t} = \mathfrak{b}$ independent of t

Introduction
The model
Arrow-Debreu equilibrium
Securities markets
Extensions
Leverage effect and real options
Large horizon
Continuous time consumption/production
Conclusion

Large horizon

- Taking the limit when $T \longrightarrow \infty$, we have

$$\bar{\theta}_t = \frac{\mathfrak{b} + \delta_t}{2\mathfrak{c} + \gamma},\ \frac{\mu_t}{\sigma_t} = \mathfrak{b},\ \frac{\sigma_t}{\bar{\theta}_t} = 2\mathfrak{c} + 1,\ \delta_\infty = \mathfrak{b}\frac{\gamma - 1}{2\mathfrak{c} + 1}$$

- $\sigma_{[0,t]} = \sqrt{\frac{1}{t}\mathrm{VAR}\left[\ln S_t\right]}$, $\mu_{[0,t]} = \frac{1}{t}\mathrm{E}[\ln S_t] + \frac{1}{2}\sigma^2_{[0,t]}$ and $\mathrm{SHARPE}_{[0,t]} = \frac{\mu_{[0,t]}}{\sigma_{[0,t]}}$
- $\bar{\sigma}_t = \mathrm{E}[\sigma_t]$, $\bar{\mu}_t = \mathrm{E}[\mu_t]$ and $\mathrm{SHARPE}_t = \frac{\mu_t}{\sigma_t} = \mathfrak{b}$ independent of t
- $\sigma_{[0,t]} \nearrow$ with t and always above $\bar{\sigma}_t$,

Large horizon

- Taking the limit when $T \longrightarrow \infty$, we have

$$\bar{\theta}_t = \frac{\mathfrak{b} + \delta_t}{2\mathfrak{c} + \gamma}, \ \frac{\mu_t}{\sigma_t} = \mathfrak{b}, \ \frac{\sigma_t}{\bar{\theta}_t} = 2\mathfrak{c} + 1, \ \delta_\infty = \mathfrak{b}\frac{\gamma - 1}{2\mathfrak{c} + 1}$$

- $\sigma_{[0,t]} = \sqrt{\frac{1}{t}\mathrm{VAR}\left[\ln S_t\right]}$, $\mu_{[0,t]} = \frac{1}{t}\mathrm{E}[\ln S_t] + \frac{1}{2}\sigma^2_{[0,t]}$ and $\mathrm{SHARPE}_{[0,t]} = \frac{\mu_{[0,t]}}{\sigma_{[0,t]}}$
- $\bar{\sigma}_t = \mathrm{E}[\sigma_t]$, $\bar{\mu}_t = \mathrm{E}[\mu_t]$ and $\mathrm{SHARPE}_t = \frac{\mu_t}{\sigma_t} = \mathfrak{b}$ independent of t
- $\sigma_{[0,t]} \nearrow$ with t and always above $\bar{\sigma}_t$,
- $\mu_{[0,t]} \nearrow$ with t and always below $\bar{\mu}_t$

Large horizon

- Taking the limit when $T \longrightarrow \infty$, we have

$$\bar{\theta}_t = \frac{\mathfrak{b} + \delta_t}{2\mathfrak{c} + \gamma}, \ \frac{\mu_t}{\sigma_t} = \mathfrak{b}, \ \frac{\sigma_t}{\bar{\theta}_t} = 2\mathfrak{c} + 1, \ \delta_\infty = \mathfrak{b}\frac{\gamma - 1}{2\mathfrak{c} + 1}$$

- $\sigma_{[0,t]} = \sqrt{\frac{1}{t}\mathrm{VAR}\left[\ln S_t\right]}, \ \mu_{[0,t]} = \frac{1}{t}\mathrm{E}[\ln S_t] + \frac{1}{2}\sigma^2_{[0,t]}$ and $\mathrm{SHARPE}_{[0,t]} = \frac{\mu_{[0,t]}}{\sigma_{[0,t]}}$
- $\bar{\sigma}_t = \mathrm{E}[\sigma_t]$, $\bar{\mu}_t = \mathrm{E}[\mu_t]$ and $\mathrm{SHARPE}_t = \frac{\mu_t}{\sigma_t} = \mathfrak{b}$ independent of t
- $\sigma_{[0,t]} \nearrow$ with t and always above $\bar{\sigma}_t$,
- $\mu_{[0,t]} \nearrow$ with t and always below $\bar{\mu}_t$
- $\mathrm{SHARPE}_{[0,t]} \searrow\nearrow$ and always lower than SHARPE_t

Belief Dispersion and Decreasing Returns

Elyès Jouini

Introduction

The model

Arrow-Debreu equilibrium

Securities markets

Extensions

Leverage effect and real options

Large horizon

Continuous time consumption/production

Conclusion

- $r_t = \rho + \mathfrak{c}\left(\frac{\mathfrak{b}+\delta_t}{2\mathfrak{c}+1}\right)^2$ procyclical and $SR_t = \frac{\mathfrak{b}-2\mathfrak{c}\delta_t}{2\mathfrak{c}+1}$ contracyclical (for $c > 0$)

- $r_t = \rho + \mathfrak{c}\left(\frac{\mathfrak{b}+\delta_t}{2\mathfrak{c}+1}\right)^2$ procyclical and $SR_t = \frac{\mathfrak{b}-2\mathfrak{c}\delta_t}{2\mathfrak{c}+1}$ contracyclical (for $c > 0$)
- $r_t = \rho + x_t$ with
  $dx_t = \kappa_t\left(\vartheta_t - x_t + \mathfrak{b}\frac{\sqrt{\mathfrak{c}}}{2\mathfrak{c}+1}\sqrt{x_t}\right)dt + \sigma_t^r\sqrt{x_t}dW_t$,
  $\kappa_t = 2\frac{\omega^2}{t\omega^2+1}$ and $\vartheta_t = \frac{1}{2}\mathfrak{c}\frac{\omega^2}{(t\omega^2+1)(2\mathfrak{c}+1)^2}$

Introduction
The model
Arrow-Debreu equilibrium
Securities markets
Extensions
Leverage effect and real options
Large horizon
Continuous time consumption/production
Conclusion

- $r_t = \rho + \mathfrak{c}\left(\frac{\mathfrak{b}+\delta_t}{2\mathfrak{c}+1}\right)^2$ procyclical and $SR_t = \frac{\mathfrak{b}-2\mathfrak{c}\delta_t}{2\mathfrak{c}+1}$ contracyclical (for $c > 0$)
- $r_t = \rho + x_t$ with
  $dx_t = \kappa_t\left(\vartheta_t - x_t + \mathfrak{b}\frac{\sqrt{\mathfrak{c}}}{2\mathfrak{c}+1}\sqrt{x_t}\right)dt + \sigma_t^r\sqrt{x_t}dW_t,$
  $\kappa_t = 2\frac{\omega^2}{t\omega^2+1}$ and $\vartheta_t = \frac{1}{2}\mathfrak{c}\frac{\omega^2}{(t\omega^2+1)(2\mathfrak{c}+1)^2}$
- For $\mathfrak{b} = 0$, shifted CIR model (CIR++, Brigo-Mercurio, 06) with time-dependent parameters

Continuous time consumption/production

Short rate and yield curve

- $r_t = \rho + \mathfrak{c}\left(\frac{\mathfrak{b}+\delta_t}{2\mathfrak{c}+1}\right)^2$ procyclical and $SR_t = \frac{\mathfrak{b}-2\mathfrak{c}\delta_t}{2\mathfrak{c}+1}$ contracyclical (for $c > 0$)
- $r_t = \rho + x_t$ with
  $dx_t = \kappa_t\left(\vartheta_t - x_t + \mathfrak{b}\frac{\sqrt{\mathfrak{c}}}{2\mathfrak{c}+1}\sqrt{x_t}\right)dt + \sigma_t^r\sqrt{x_t}dW_t,$
  $\kappa_t = 2\frac{\omega^2}{t\omega^2+1}$ and $\vartheta_t = \frac{1}{2}\mathfrak{c}\frac{\omega^2}{(t\omega^2+1)(2\mathfrak{c}+1)^2}$
- For $\mathfrak{b} = 0$, shifted CIR model (CIR++, Brigo-Mercurio, 06) with time-dependent parameters
- For $\rho = 0$, CIR model with time-dependent parameters

Continuous time consumption/production

Short rate and yield curve

- $r_t = \rho + \mathfrak{c}\left(\frac{\mathfrak{b}+\delta_t}{2\mathfrak{c}+1}\right)^2$ procyclical and $SR_t = \frac{\mathfrak{b}-2\mathfrak{c}\delta_t}{2\mathfrak{c}+1}$ contracyclical (for $c > 0$)
- $r_t = \rho + x_t$ with
  $dx_t = \kappa_t\left(\vartheta_t - x_t + \mathfrak{b}\frac{\sqrt{\mathfrak{c}}}{2\mathfrak{c}+1}\sqrt{x_t}\right)dt + \sigma_t^r\sqrt{x_t}dW_t,$
  $\kappa_t = 2\frac{\omega^2}{t\omega^2+1}$ and $\vartheta_t = \frac{1}{2}\mathfrak{c}\frac{\omega^2}{(t\omega^2+1)(2\mathfrak{c}+1)^2}$
- For $\mathfrak{b} = 0$, shifted CIR model (CIR++, Brigo-Mercurio, 06) with time-dependent parameters
- For $\rho = 0$, CIR model with time-dependent parameters
- Generate a rich class of shapes for the yield curve (but always increasing in the short-run)

Conclusion

- We have attempted to link classical financial asset pricing models to the pricing, output, and investment models in the economics literature

Conclusion

- We have attempted to link classical financial asset pricing models to the pricing, output, and investment models in the economics literature
- The firm maximizes its market value, its decisions are impacted by financial markets, and financial markets are impacted by its decisions in return

Conclusion

- We have attempted to link classical financial asset pricing models to the pricing, output, and investment models in the economics literature
- The firm maximizes its market value, its decisions are impacted by financial markets, and financial markets are impacted by its decisions in return
- We retrieved some empirical regularities and derived testable consequences

Conclusion

Introduction
The model
Arrow-Debreu equilibrium
Securities markets
Extensions
Leverage effect and real options
Large horizon
Continuous time consumption/production
Conclusion

- We have attempted to link classical financial asset pricing models to the pricing, output, and investment models in the economics literature
- The firm maximizes its market value, its decisions are impacted by financial markets, and financial markets are impacted by its decisions in return
- We retrieved some empirical regularities and derived testable consequences
- It may be of interest to explore

Conclusion

- We have attempted to link classical financial asset pricing models to the pricing, output, and investment models in the economics literature
- The firm maximizes its market value, its decisions are impacted by financial markets, and financial markets are impacted by its decisions in return
- We retrieved some empirical regularities and derived testable consequences
- It may be of interest to explore
  - from the financial point of view, debt as a strategic variable

Conclusion

Introduction
The model
Arrow-Debreu equilibrium
Securities markets
Extensions
Leverage effect and real options
Large horizon
Continuous time consumption/production
Conclusion

- We have attempted to link classical financial asset pricing models to the pricing, output, and investment models in the economics literature
- The firm maximizes its market value, its decisions are impacted by financial markets, and financial markets are impacted by its decisions in return
- We retrieved some empirical regularities and derived testable consequences
- It may be of interest to explore
  - from the financial point of view, debt as a strategic variable
  - from the corporate governance point of view, delegation of firm's decisions and its impact

Conclusion

- We have attempted to link classical financial asset pricing models to the pricing, output, and investment models in the economics literature
- The firm maximizes its market value, its decisions are impacted by financial markets, and financial markets are impacted by its decisions in return
- We retrieved some empirical regularities and derived testable consequences
- It may be of interest to explore
  - from the financial point of view, debt as a strategic variable
  - from the corporate governance point of view, delegation of firm's decisions and its impact
  - from a macroeconomic point of view, impact of real or perceived technological changes (e.g., dot-com bubble, data economy, Uberization, etc.) on volatilities